# FAMILIES OF CANONICALLY POLARIZED VARIETIES OVER SURFACES

STEFAN KEBEKUS AND SÁNDOR J. KOVÁCS

ABSTRACT. Shafarevich's hyperbolicity conjecture asserts that a family of curves over a quasi-projective 1-dimensional base is isotrivial unless the logarithmic Kodaira dimension of the base is positive. More generally it has been conjectured by Viehweg that the base of a smooth family of canonically polarized varieties is of log general type if the family is of maximal variation. In this paper, we relate the variation of a family to the logarithmic Kodaira dimension of the base and give an affirmative answer to Viehweg's conjecture for families parametrized by surfaces.

## CONTENTS



## 1. INTRODUCTION

Let $B^\circ$ be a smooth quasi-projective complex curve and $q > 1$ a positive integer. Shafarevich conjectured [Sha63] that the set of non-isotrivial families of smooth projective curves of genus $q$ over $B^\circ$ is finite. Shafarevich further conjectured that if the logarithmic Kodaira dimension, for a definition see below, satisfies $\kappa(B^\circ) \leq 0$, then no such families exist. This conjecture, which later played an important role in Faltings' proof of the Mordell conjecture, was confirmed by Parshin [Par68] for $B^\circ$ projective and by Arakelov [Ara71] in general. We refer the reader to the survey articles [Vie01] and [Kov03] for a historical overview and references to related results.

It is a natural and important question whether similar statements hold for families of higher dimensional varieties over higher dimensional bases. Families over a curve have been studied by several authors in recent years and they are now fairly well understood—the strongest results known were obtained in [VZ01, VZ02], and [Kov02]. For higher

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dimensional bases, however, a complete picture is still missing and subvarieties of the corresponding moduli stacks are not well understood. As a first step toward a better understanding, Viehweg proposed the following:

**Conjecture 1.1.** [Vie01, 6.3]. *Let $f^\circ : X^\circ \to S^\circ$ be a smooth family of canonically polarized varieties. If $f^\circ$ is of maximal variation, then $S^\circ$ is of log general type.*

We briefly recall the relevant definitions, as they will also be important in the statement of our main result. The first is the variation, which measures the birational non-isotriviality of a family.

**Definition 1.2.** *Let $f : X \to S$ be a projective family over an irreducible base $S$ defined over an algebraically closed field $k$ and let $\overline{k(S)}$ denote the algebraic closure of the function field of $S$. The variation of $f$, denoted by $\operatorname{Var} f$, is defined as the smallest integer $\nu$ for which there exists a subfield $K$ of $\overline{k(S)}$, finitely generated of transcendence degree $\nu$ over $k$ and a $K$-variety $F$ such that $X \times_S \operatorname{Spec} \overline{k(S)}$ is birationally equivalent to $F \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{k(S)}$.*

*Remark* 1.2.1. In the setup of Definition 1.2, if the fibers are canonically polarized complex varieties, moduli schemes are known to exist, and the variation is the same as either the dimension of the image of $S$ in moduli, or the rank of the Kodaira-Spencer map at the general point of $S$.

**Definition 1.3.** *Let $S^\circ$ be a smooth quasi-projective variety and $S$ a smooth projective compactification of $S^\circ$ such that $D := S \setminus S^\circ$ is a divisor with simple normal crossings. The logarithmic Kodaira dimension of $S^\circ$, denoted by $\kappa(S^\circ)$, is defined to be the Kodaira-Iitaka dimension, $\kappa(S, D)$, of the line bundle $\mathcal{O}_S(K_S + D) \in \operatorname{Pic}(S)$. The variety $S^\circ$ is called of* log general type *if $\kappa(S^\circ) = \dim S^\circ$, i.e., the divisor $K_S + D$ is big.*

*Remark* 1.3.1. It is a standard fact in logarithmic geometry that a compactification $S$ with the described properties exists, and that the logarithmic Kodaira dimension $\kappa(S^\circ)$ does not depend on the choice of the compactification $S$.

1.A. **Statement of the main result.** Our main result describes families of canonically polarized varieties over quasi-projective surfaces. We relate the variation of the family to the logarithmic Kodaira dimension of the base and give an affirmative answer to Viehweg's Conjecture 1.1 for families over surfaces.

**Theorem 1.4.** *Let $S^\circ$ be a smooth quasi-projective complex surface and $f^\circ : X^\circ \to S^\circ$ a smooth non-isotrivial family of canonically polarized complex varieties. Then the following holds.*

(1.4.1) *If $\kappa(S^\circ) = -\infty$, then $\operatorname{Var}(f^\circ) \leq 1$.*
(1.4.2) *If $\kappa(S^\circ) \geq 0$, then $\operatorname{Var}(f^\circ) \leq \kappa(S^\circ)$.*

*In particular, Viehweg's Conjecture holds for families over surfaces,*

For the special case of $\kappa(S^\circ) = 0$, this statement was proved by Kovács [Kov97, 0.1] when $S^\circ$ is an abelian variety and more generally by Viehweg and Zuo [VZ02, 5.2] when $T_S(-\log D)$ is weakly positive.

A slightly weaker statement holds for families of minimal varieties, see Section 8 below. In a forthcoming paper we will give a more precise geometric description of $f^\circ$ in the case of $\kappa(S^\circ) \leq 1$.

*Remark* 1.5. Notice that in the case of $\kappa(S^\circ) = -\infty$ one cannot expect a stronger statement. For an easy example take any non-isotrivial smooth family of canonically polarized varieties over a curve $g : Z \to C$, set $X := Z \times \mathbb{P}^1$, $S^\circ := C \times \mathbb{P}^1$, and let $f^\circ := g \times \mathrm{id}_{\mathbb{P}^1}$ be the obvious morphism. Then we clearly have $\kappa(S^\circ) = -\infty$ and $\operatorname{Var}(f) = 1$.

In view of Theorem 1.4, we propose the following generalization of Viehweg's conjecture.

**Conjecture 1.6.** *Let $f^\circ : X^\circ \to S^\circ$ be a smooth family of canonically polarized varieties. Then either $\kappa(S^\circ) = -\infty$ and $\operatorname{Var}(f^\circ) < \dim S^\circ$, or $\operatorname{Var}(f^\circ) \leq \kappa(S^\circ)$.*

1.B. **Outline of the paper.** Throughout the paper we work over $\mathbb{C}$, the field of complex numbers.

The paper is divided into two parts. In the first part comprising Sections 2 and 3 we recall and establish techniques that might be of independent interest. Section 2 summarizes results in logarithmic geometry and logarithmic deformation theory. In Section 3 we consider logarithmic pairs $(S, D)$ where $S$ is a birationally ruled surface, and construct a sequence of blowing down $(-1)$-curves that can be used to simplify the self-intersection graph of the boundary $D$.

In the second part of the paper we employ these techniques in order to prove Theorem 1.4. After the notation is set up in Section 4, we consider the cases where the logarithmic Kodaira dimension of $S^\circ$ is 1, 0 or $-\infty$ in Sections 5–7, respectively.

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# PART 1. **TECHNIQUES**

## 2. LOGARITHMIC GEOMETRY

Throughout the current section, let $S$ be a smooth projective variety and $D \subset S$ a reduced divisor with simple normal crossings. As follows, we recall a number of facts concerning this setup and include proofs wherever we could not find an adequate reference.

2.A. **The Logarithmic Minimal Model Program.** If $S$ is a surface and the logarithmic Kodaira dimension $\kappa(K_S + D)$ is non-negative, we will frequently need to consider the $(S, D)$-logarithmic minimal model program, which is briefly recalled here. The reader is referred to [KM98] for the relevant definitions, for proofs and for a full discussion.

*Fact* 2.1 (Logarithmic Minimal Model Program, [KM98, (3.47)]). If $\dim S = 2$ and $\kappa(K_S + D) \geq 0$, there exists a birational morphism $\phi : S \to S_\lambda$ from $S$ to a normal surface $S_\lambda$ such that

(2.1.1) The morphism $\phi$ is the composition of finitely many log contractions.

(2.1.2) If we set $D_\lambda := \phi(D)$ to be the cycle-theoretic image divisor, then
- (a) The pair $(S_\lambda, D_\lambda)$ has only *dlt* singularities and $S_\lambda$ itself is $\mathbb{Q}$-factorial [KM98, (3.36), (3.44)]. In particular, $S_\lambda$ has only quotient singularities.
- (b) The log canonical divisor $K_{S_\lambda} + D_\lambda$ is nef.
- (c) The log Kodaira dimension remains unchanged,

$$\kappa(K_{S_\lambda} + D_\lambda) = \kappa(K_S + D).$$

*Remark* 2.2. In the discussion of the minimal model program one needs to consider several classes of singularities. The large number of notions, and the fact that the definitions found in the literature are not always obviously equivalent makes the field somewhat difficult to navigate for the outsider. For the reader's convenience, we briefly indicate how that fact that $S_\lambda$ has only quotient singularities follows from the assumption that $(S_\lambda, D_\lambda)$ has only $\mathbb{Q}$-factorial *dlt* singularities:

By [KM98, (2.42)], if $x \in S_\lambda$ is any point, then either $x$ is a smooth point of $S_\lambda$, or $(S_\lambda, D_\lambda)$ is *plt* at $x$. We can thus assume without loss of generality that $(S_\lambda, D_\lambda)$ is *plt* everywhere, i.e. that

$$\operatorname{discrep}(S_\lambda, D_\lambda) > -1.$$

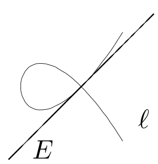
By [KM98, (2.27)], $\operatorname{discrep}(S_\lambda, 0) \geq \operatorname{discrep}(S_\lambda, D_\lambda) > -1$. By definition, [KM98, (2.34)], this means that $S_\lambda$ is *log terminal*. The classification of *log terminal* surface singularities, [KM98, (4.18)] then gives the claim.

*Remark* 2.3. We remark that the support of $D_\lambda$ is generally *not* equal to the image $\phi(D)$, as it may well happen that $\phi(D)$ contains isolated points which do not appear in the cycle-theoretic image. This observation will later become important in Section 6.C and in the proof of Proposition 6.6.

*Fact* 2.4 (Logarithmic Abundance Theorem in Dimension 2, [KM98, (3.3)]). The linear system $|n(K_{S_\lambda} + D_\lambda)|$ is basepoint-free for sufficiently large and divisible $n \in \mathbb{N}$.

2.B. **Logarithmic deformation theory.** In Sections 6 and 7 we will have to deal with families of curves on $S$ that intersect the boundary divisor $D$ in one or two points. In counting these points, intersection multiplicity does not play any role, but the number of local analytic branches of the curves does. More precisely, we use the following definition.

**Definition 2.5.** *Let $X$ be an algebraic variety, $E \subset X$ an algebraic set, and $\ell \subset X$ a reduced proper curve with normalization $\nu : \widetilde{\ell} \to \ell$. We say that "$\ell$ intersects $E$ in $d$ points" if the preimage $\nu^{-1}(E)$ is supported on exactly $d$ closed points of $\widetilde{\ell}$.*




In the sense of Definition 2.5, $\ell$ intersects $E$ in two points.

FIGURE 2.1. Number of intersection points

*Remark* 2.6. Suppose we are given a proper birational morphism $\phi : X \to X'$, an algebraic set $E \subset X$ and a family of curves $\ell'_t \subset X'$ that intersect $\phi(E)$ in exactly $d$ points. Assume further that none of the $\ell'_t$ is contained in the set of fundamental points of $\phi^{-1}$. Then the strict transforms give a (possibly disconnected) family $\ell_t$ of curves on $X$ that intersect $E$ in no more than $d$ points. If $E$ contains the $\phi$-exceptional locus, then the strict transforms intersect $E$ in exactly $d$ points.

For our applications, we need to consider a family $\ell_t$ of rational curves in $S$ that intersect $D$ in two points. Our aim in this section is to discuss an algebraic parameter space for such curves. The construction is based on the observation that for any such curve $\ell_t$ there exists a morphism $\nu_t : \mathbb{P}^1 \to \ell_t \subset S$ such that $\nu_t^{-1}(D)$ is supported exactly on the points $x_0 := [0:1]$ and $x_\infty := [1:0]$. Therefore, it makes sense to consider the space

$$\mathcal{H} := \left\{ f \in \operatorname{Hom}_{\mathrm{bir}}(\mathbb{P}^1, S) \mid f^{-1}(D) \text{ is supported exactly on } x_0 \text{ and } x_\infty \right\}$$

with the obvious structure as a closed, but possibly non-reduced subscheme of $\operatorname{Hom}_{\mathrm{bir}}(\mathbb{P}^1, S)$, the space of generically injective morphisms $\mathbb{P}^1 \to S$.

The space $\mathcal{H}$, and its infinitesimal structure has been studied in [KMcK99] using a slightly different language. We recall some of their results here and include proofs wherever we had difficulties to follow the original arguments.

**Proposition 2.7.** *If $f \in \mathcal{H}$ is any closed point, then the Zariski tangent space to $\mathcal{H}$ at $f$ is canonically isomorphic to*

$$T_{\mathcal{H}}|_f \simeq H^0\big(\mathbb{P}^1, f^*(T_S(-\log D))\big).$$

*Proof.* Let $\mathscr{D} = \{D_1, \ldots, D_n\}$ be the set of irreducible components of $D$, $E_i := f^*(D_i)$ the associated Cartier divisors on $\mathbb{P}^1$, and let $\mathscr{E} = \{E_1, \ldots, E_n\}$. Notice that all of the $E_i$ are supported on $x_0$ and $x_\infty$. If $\mathcal{H}_f \subset \mathcal{H}$ is the connected component that contains $f$,

then $\mathfrak{H} = \mathrm{Hom}(\mathbb{P}^1, S, \mathscr{E} \subset \mathscr{D})$, defined in [KMcK99, Sect. 5], is a subscheme of $\mathcal{H}$ that contains $\mathcal{H}_f$. Hence, the claim follows from [KMcK99, (5.3)]. $\square$

It is well known in the theory of rational curves on algebraic varieties that if $S$ is a uniruled manifold and $\ell$ is a rational curve that passes through a very general point of $S$, then $\ell$ is free, and its deformations avoid any given subset of codimension $\geq 2$. More precisely, for any given subset $E \subset S$ with $\mathrm{codim}_S E \geq 2$ there exists a deformation $\ell'$ of $\ell$ that does not intersect $E$. We show that a similar property holds for $\mathcal{H}$.

**Proposition 2.8** (Small Set Avoidance)**.** *Let $\mathcal{H}' \subset \mathcal{H}$ be an irreducible component such that the associated curves dominate $S$. If $M \subset S \setminus D$ is any closed set of $\mathrm{codim}_S M \geq 2$, then there exists a non-empty open set $\mathcal{H}'_0 \subset \mathcal{H}'$ such that for all $f' \in \mathcal{H}'_0$ the image does not intersect $M$, i.e.*

$$M \cap f'(\mathbb{P}^1) = \emptyset.$$

The proof of Proposition 2.8, which we give on page 7 at the end of this section is based on a number of results that we prove first. We start with an estimate for the dimension of $\mathcal{H}$ that we formulate and prove in the next two lemmata.

**Definition 2.9.** *Let $\pi : X \to Y$ be a finite surjective morphism of degree $d$. The* set-theoretic branch locus *of $\pi$ is the set of points in $Y$ whose set-theoretic preimage contains strictly less than $d$ points.*

**Lemma 2.10.** *Let $H$ be an irreducible variety and $D \subset \mathbb{P}^1 \times H$ an irreducible subvariety such that $\pi_2|_D : D \to H$ is a finite surjective morphism of degree $d$ with set-theoretic branch locus $B$. Then either $B = \emptyset$, or $B$ is a closed subvariety of pure codimension 1.*

*Proof.* Performing a base change, if necessary, we can assume without loss of generality that $H$ is normal. The variety $D$ is then a well-defined family of algebraic cycles in the sense of [Kol96, I.3.10], and therefore yields a morphism

$$\gamma : H \to \mathrm{Chow}^d(\mathbb{P}^1) = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \big/ \text{permutation} \simeq \mathbb{P}(\mathrm{Sym}^d \mathbb{A}^2).$$

If $\Delta \subset \mathbb{P}(\mathrm{Sym}^d \mathbb{A}^2)$ is the discriminant divisor, i.e. the branch locus of the morphism

$$\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \big/ \text{permutation} \simeq \mathbb{P}(\mathrm{Sym}^d \mathbb{A}^2),$$

then the morphism $D \to H$ is branched at a point $\eta \in H$ iff $\gamma(\eta) \in \Delta$. But since $\Delta \subset \mathbb{P}(\mathrm{Sym}^d \mathbb{A}^2)$ is Cartier, the claim follows. $\square$

The proof of Lemma 2.10 shows, after passing to the normalization, that the branch locus $B$ is even a Cartier-divisor, but we will not need this observation here. The proposed estimate for the dimension of $\mathcal{H}$ then goes as follows.

**Lemma 2.11** ([KMcK99, 5.1, 5.3])**.** *If $\eta \in \mathcal{H}$ is any point, then*

$$\dim_\eta \mathcal{H} \geq \dim_\eta \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, S) - \underbrace{\deg_{\mathbb{P}^1} \eta^*(\mathcal{O}_S(D))}_{=:d}.$$

*Proof.* Let $H \subset \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, S)$ be an irreducible component through $\eta$ which is of maximal dimension. We will prove Lemma 2.11 by an inductive construction of a subvariety that contains $\eta$, satisfies the dimension bound, and is contained in $\mathcal{H}$. More precisely, we claim the following.

*Claim* 2.11.1. There exists a sequence of subvarieties

$$H = H^{(0)} \supset H^{(1)} \supset \cdots \supset H^{(d-2)} \ni \eta$$

such that $\mathrm{codim}_H H^{(i)} = i$, and such that for general closed points $f^{(i)} \in H^{(i)}$, we have $\#\left(f^{(i)}\right)^{-1}(D) \leq d - i$

*Proof.* We prove the claim inductively, using the index $i$ of $H^{(i)}$. To start the induction, consider $i = 0$. It is clear from intersection theory that if $f^{(0)} \in H$ is a general closed point, then $\#\big(f^{(0)}\big)^{-1}(D) \leq d$. For the inductive step, assume that the subvariety $H^{(i)}$ is already constructed. Consider the universal morphism $\mu_i : \mathbb{P}^1 \times H^{(i)} \to S$ and the reduced preimage

$$D^{(i)} := \mu_i^{-1}(D)_{\mathrm{red}} \subset \mathbb{P}^1 \times H^{(i)}.$$

By induction there exists an open set $\eta \in H_\circ^{(i)} \subseteq H^{(i)}$ such that $D_\circ^{(i)} = D^{(i)} \cap \pi_2^{-1} H_\circ^{(i)}$ surjects finitely onto $H_\circ^{(i)}$ with at most $d - i$ sheets. Observe, that as long as $d - i > 2$, $\eta$ will be in the set-theoretic branch locus of $\pi_2|_{D_\circ^{(i)}} : D_\circ^{(i)} \to H_\circ^{(i)}$. If we set

$$H^{(i+1)} := \text{closure of one component of the set-theoretic branch locus of } \pi_2|_{D_\circ^{(i)}},$$

then by Lemma 2.10 $\dim H^{(i+1)} = \dim H^{(i)} - 1$, and a general point of $H^{(i+1)}$ has at most $d - (i + 1)$ preimages on $D^{(i)}$. Claim 2.11.1 then follows. $\square$

According to Claim 2.11.1 there exists a subvariety containing $\eta$, $H^{(d-2)} \subseteq H$, of dimension $\dim H^{(d-2)} = \dim_\eta H - d + 2$. Since $\mu_{d-2}^{-1}(D)$ is a Cartier divisor on $\mathbb{P}^1 \times H^{(d-2)}$, the non-empty subvarieties

$$H_0^{(d-2)} := \pi_2\left(\mu_{d-2}^{-1}(D)_{\mathrm{red}} \cap \{x_0\} \times H^{(d-2)}\right) \subseteq H^{(d-2)}, \text{ and}$$
$$H_{0,\infty}^{(d-2)} := \pi_2\left(\mu_{d-2}^{-1}(D)_{\mathrm{red}} \cap \{x_\infty\} \times H_0^{(d-2)}\right) \subseteq H_0^{(d-2)}$$

each contain $\eta$ and have codimension at most 1 in one another. In other words, we have

$$\dim H_{0,\infty}^{(d-2)} \geq \dim H^{(d-2)} - 2 = \dim_\eta H - d. \tag{2.11.2}$$

It follows from Claim 2.11.1 that for all closed points $f \in H^{(d-2)}$, the associated morphism $f : \mathbb{P}^1 \to S$ satisfies $\#f^{-1}(D) \leq 2$. Because $f$ is contained in $H_{0,\infty}^{(d-2)} \subseteq H_0^{(d-2)}$, we also have $f(x_0) \in D$ and $f(x_\infty) \in D$, respectively. In summary, we have seen that $H_{0,\infty}^{(d-2)} \subseteq \mathcal{H}$, which combined with (2.11.2) proves Lemma 2.11. $\square$

We note that a more detailed analysis of the construction could be used to show that $\mathcal{H}$ is a local complete intersection. To continue the preparation for the proof of Proposition 2.8 we discuss the pull-back of the logarithmic tangent sheaf via a general morphism in $\mathcal{H}'$.

**Lemma 2.12.** *Under the assumptions of Proposition 2.8, let $f \in \mathcal{H}'$ be a general element. Then $f^*\big(T_S(-\log D)\big)$ is globally generated on $\mathbb{P}^1$.*

*Proof.* Set $n := \dim S$. Working on $\mathbb{P}^1$, it suffices to show that $f^*(T_S(-\log D))$ is generated by global sections at a general point $y \in \mathbb{P}^1$, i.e., that there exist sections $\sigma_1, \ldots \sigma_n \in H^0(\mathbb{P}^1, f^*(T_S(-\log D)))$ that are linearly independent at $y$.

To construct $\sigma_1$, observe that the natural action of $\mathbb{C}^*$ on $\mathbb{P}^1$ that fixes $x_0 = [0 : 1]$ and $x_\infty = [1 : 0]$ yields a non-trivial deformation of $f$ in $\mathcal{H}$. Let $\sigma_1$ be an associated infinitesimal deformation which, by general choice of $y$, does not vanish at $y$.

In order to find $\sigma_2, \ldots, \sigma_n$, observe that the curves associated with $\mathcal{H}$ dominate $S$. By general choice of $f$, we can therefore assume that the universal morphism

$$\mu : \mathbb{P}^1 \times \mathcal{H} \to S$$

has rank $n$ at $(y, f)$. The description [Kol96, II.3.4] of the tangent morphism $T\mu$ then yields the existence of infinitesimal deformations $\sigma_2, \ldots, \sigma_n$ whose evaluations $\sigma_i(y)$ along with $\sigma_1(y)$ are linearly independent and not tangent to the image of $f$. $\square$

**Corollary 2.13.** *Under the conditions of Lemma 2.12, both $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, S)$ and $\mathcal{H}'$ are reduced and smooth at the point $f$.*

*Proof.* Lemma 2.12 implies that $f^*(T_S)$ is also globally generated on $\mathbb{P}^1$ since it contains the globally generated locally free subsheaf $f^*(T_S(-\log D))$ of the same rank. Then $H^1(\mathbb{P}^1, f^*(T_S)) = 0$, so $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, S)$ is reduced and smooth of dimension $h^0(\mathbb{P}^1, f^*(T_S))$ by [Kol96, I.2.16]. This, combined with Lemma 2.11 implies that

$$h^0(\mathbb{P}^1, f^*(T_S)) - d \le \dim_f \mathcal{H}' \le \dim T_{\mathcal{H}'}|_f. \tag{2.13.1}$$

Proposition 2.7 and the fact that $\deg_{\mathbb{P}^1} f^*(T_S) = \deg_{\mathbb{P}^1} f^*(T_S(-\log D)) + d$ imply that

$$h^0(\mathbb{P}^1, f^*(T_S)) - d = h^0(\mathbb{P}^1, f^*(T_S(-\log D))) = \dim T_{\mathcal{H}'}|_f, \tag{2.13.2}$$

The (in)equalities (2.13.1) and (2.13.2) together imply that $\dim T_{\mathcal{H}'}|_f = \dim_f \mathcal{H}'$. Therefore, we obtain that $\mathcal{H}'$ is reduced and smooth at the point $f$. □

*Proof of Proposition 2.8.* Consider the standard diagram

$$\begin{tikzcd} \mathbb{P}^1 \times \mathcal{H}' \arrow[r, "\mu"', "\text{univ. morphism}"'] \arrow[d, "\pi"', "\text{projection}"] & S \\ \mathcal{H}', & \end{tikzcd}$$

and let $\mathcal{M} := \big(\mu^{-1}(M)\big)_{\mathrm{red}}$ be the set-theoretic preimage of $M$ via $\mu$. Since $\pi$ is proper, it is enough to prove that $\pi(\mathcal{M}) \neq \mathcal{H}'$. Assume to the contrary, i.e., assume that $\mathcal{M}$ surjects onto $\mathcal{H}'$ and choose a point $y \in (\pi|_{\mathcal{M}})^{-1}(f)$. Since $\mathcal{H}'$ is smooth at $f$, the general choice of $f$ implies that $\pi|_{\mathcal{M}}$ is étale at $(y, f)$. Then the global generation of $f^*\big(T_S(-\log D)\big)$ and the standard description of the tangent morphism $T\mu$, [Kol96, II.3.4], yield that the rank of $T\mu|_{\mathcal{M}}$ at $(y, f)$ is at least $n-1$. In particular, $\operatorname{codim}_S M \le 1$, a contradiction. □

2.C. **Logarithmic differentials.** Throughout the proof of the main theorem we need to use the sheaf $\Omega^1_S(\log D)$ of 1-forms with logarithmic poles along $D$. For the definition and detailed discussion of this notion the reader is referred to either [Del70, Chap. 3] or [EV92, § 2]. We will need to describe $\Omega^1_S(\log D)$ in terms of its restriction to curves in $S$.

**Lemma 2.14.** *Let $F \subset S$ be a smooth curve that intersects $D$ transversally. Then the restriction $\Omega^1_S(\log D)|_F$ is an extension of line bundles, as follows:*

$$0 \to N^\vee_{F/S} \to \Omega^1_S(\log D)|_F \to \Omega^1_F(\log D|_F) \to 0. \tag{2.14.1}$$

*If $D = \sum_{i=1}^r D_i$ is the decomposition of $D$ to irreducible components, then the restriction $\Omega^1_S(\log D)|_{D_1}$ is an extension of line bundles, as follows:*

$$0 \to \Omega^1_{D_1}(\log(D - D_1)|_{D_1}) \to \Omega^1_S(\log D)|_{D_1} \to \mathcal{O}_{D_1} \to 0. \tag{2.14.2}$$

*Furthermore, if $\dim S = 2$, then*

$$\Omega^1_{D_1}(\log(D - D_1)|_{D_1}) \simeq \Omega^1_{D_1} \otimes \mathcal{O}_{D_1}\big((D - D_1)|_{D_1}\big). \tag{2.14.3}$$

*Proof.* To prove (2.14.1), consider the following diagram with exact rows [EV92, 2.3a]:

$$\begin{tikzcd} 0 \arrow[r] & \Omega^1_S|_F \arrow[r] \arrow[d, "\theta_1"] & \Omega^1_S(\log D)|_F \arrow[r] \arrow[d, "\theta_2"] & \bigoplus_{i=1}^r \mathcal{O}_{D_i}|_F \arrow[r] \arrow[d, "\theta_3"] & 0 \\ 0 \arrow[r] & \Omega^1_F \arrow[r] & \Omega^1_F(\log D|_F) \arrow[r] & \bigoplus_{i=1}^r \mathcal{O}_{D_i|_F} \arrow[r] & 0. \end{tikzcd}$$

Notice that $\theta_3$ is an isomorphism and $\theta_1$ is surjective, and hence it follows from the Snake Lemma that $\theta_2$ is also surjective and $\ker\theta_2 \simeq \ker\theta_1 \simeq N^\vee_{F/S}$. This shows (2.14.1).

To prove (2.14.2), consider the following diagram with exact rows [EV92, 2.3c]:

$$\begin{tikzcd}
\Omega^1_S(\log D)(-D_1) \arrow[r, hook] \arrow[d, "\vartheta_1"] & \Omega^1_S(\log(D-D_1)) \arrow[r, two heads] \arrow[d, "\vartheta_2"] & \Omega^1_{D_1}(\log(D-D_1)|_{D_1}) \arrow[d, dotted, "\vartheta_3"] \\
\Omega^1_S(\log D)(-D_1) \arrow[r, hook] & \Omega^1_S(\log D) \arrow[r, two heads] & \Omega^1_S(\log D)|_{D_1}.
\end{tikzcd}$$

Since the rows are exact, the morphisms $\vartheta_1$ and $\vartheta_2$ imply the existence of $\vartheta_3$. Observe that $\vartheta_1$ is an isomorphism and $\vartheta_2$ is injective, hence it follows from the Snake Lemma that $\vartheta_3$ is also injective and $\operatorname{coker}\vartheta_3 \simeq \operatorname{coker}\vartheta_2 \simeq \mathcal{O}_{D_1}$ [EV92, 2.3b].

Finally, if $\dim S = 2$, then $D_1$ is a smooth curve, and (2.14.3) follows immediately from the definition. □

## 3. Controlled minimal models of birationally ruled surfaces

In this section, we consider log pairs $(S, D)$, where $S$ is a birationally ruled surface whose boundary intersects the ruling with multiplicity two. More precisely, we make the following assumption throughout the present section.

**Assumption 3.1.** *Let $S$ be a smooth projective surface and $D \subset S$ a simple normal crossing divisor. Assume that there exists a morphism $\pi : S \to C$ whose general fiber is isomorphic to $\mathbb{P}^1$. If $t \in C$ is any point, set $S_t := \pi^{-1}(t)$ and assume that $D \cdot S_t = 2$.*

Our principle aim in this section is to relate the logarithmic Kodaira dimension $\kappa(S \setminus D)$ with the genus of the base curve $C$ and with the number and type of fiber components contained in $D$.

The relation in question is formulated in Propositions 3.5 and 3.6 using a certain sequence of blowing down vertical $(-1)$-curves which simplifies the self-intersection graph of $D$ and eventually leads to a $\mathbb{P}^1$-bundle over $C$. The construction of this sequence is explained in Section 3.A below.

### 3.A. Construction of a minimal model. Setup of notation.

To describe the sequence of blowings down we use the following terminology.

*Notation* 3.2. A curve $E \subset S$ is called *vertical* if it maps to a point in $C$. Let $D = D^h + D^v$ be the associated decomposition of the divisor $D$, where $D^v$ is the sum of the vertical components, and $D^h$ the components that surject onto $C$.

Now consider the sequence of blowings down of vertical $(-1)$-curves, as given by Algorithm 3.3 on page 9 below. The construction obviously depends on choices and is therefore not unique. While the results stated in section 3.B are independent of the choices made, we fix a particular set of choices for the remainder of the section and do not pursue the uniqueness question further.

*Notation* 3.4. We fix a set of choices, set $S = S_0$ and denote the morphisms that occur in Algorithm 3.3 as follows.

$$\begin{tikzcd}
S_0 \arrow[r, "\beta_0\text{: blow-down}"'] \arrow[rr, bend left, "\rho_i"] & S_1 \cdots \arrow[r, "\beta_{i-1}\text{: blow-down}"'] & S_i \arrow[r, "\beta_i\text{: blow-down}"'] \arrow[rr, bend left, "\pi_i"] & \cdots S_m \arrow[r, "\pi_m", "\mathbb{P}^1-\text{bundle}"'] & C.
\end{tikzcd}$$

**Algorithm 3.3**: Construction of a good relative minimal model of $S$

```
Step 0:  Setup
```

$i := 0, \quad S_0 := S, \quad D_0^h := D^h, \quad D_0^v := D^v$

`Step 1:  blow down curves that are disjoint from` $D^h$

**while** *there exists a vertical* $(-1)$*-curve* $E_i \subset S_i$*, disjoint from* $D_i^h$ **do**

- $S_{i+1}$ := blow-down of $S_i$ along $E_i$
- $D_{i+1}^h$ := cycle-theoretic image of $D^h$ in $S_{i+1}$
- $D_{i+1}^v$ := cycle-theoretic image of $D^v$ in $S_{i+1}$
- $i \leftarrow i+1$

$\boxed{k_1 := i}$

`Step 2:  for each reducible fiber` $F$ `blow down` $(-1)$`-curves contained in` $F$`, always taking curves in` $D^v$ `if possible. Stop if` $D_i^h$ `and` $D_i^v$ `no longer intersect in` $F$.

**for** *each reducible fiber* $F \subset S_i$ **do**

- **while** $D_i^h \cap D_i^v \cap F \neq \emptyset$ *and there exists a* $(-1)$*-curve in* $F$ **do**
  - **if** *there exists a vertical* $(-1)$*-curve in* $F$*, contained in* $D_i^v$ **then**
    - $E_i$ := a vertical $(-1)$-curve in $F$, contained in $D_i^v$
  - **else**
    - $E_i$ := any vertical $(-1)$-curve in $F$
  - $S_{i+1}$ := blow-down of $S_i$ along $E_i$
  - $D_{i+1}^h$ := cycle-theoretic image of $D^h$ in $S_{i+1}$
  - $D_{i+1}^v$ := cycle-theoretic image of $D^v$ in $S_{i+1}$
  - $i \leftarrow i+1$

$\boxed{k_2 := i}$

`Step 3:  blow down the remaining vertical` $(-1)$`-curves`

**while** *there exists a vertical* $(-1)$*-curve* $E_i \subset S_i$ **do**

- $S_{i+1}$ := blow-down of $S_i$ along $E_i$
- $i \leftarrow i+1$

$\boxed{m := i}$

`Now` $S_i$ `does not contain any vertical` $(-1)$`-curve, and is therefore relatively minimal over` $C$.

If $t \in C$ is any point, let $S_t := \pi^{-1}(t)$ and $S_{i,t} := \pi_i^{-1}(t)$ be the scheme-theoretic fibers. In addition, we will also consider the following objects.

| | | |
|---|---|---|
| $k_1, k_2, m$ | ... | the indexes marking the end of Steps 1, 2, and 3 in Algorithm 3.3 |
| $E_i$ | ... | the $\beta_i$-exceptional vertical $(-1)$-curve in $S_i$ |
| $D_i^h, D_i^v$ | ... | the cycle-theoretic images of $D^h$ and $D^v$ in $S_i$, respectively |
| $C_0$ | ... | the section of $\pi_m$ with minimal self-intersection number |
| $F_m$ | ... | the numerical class of a fiber of $\pi_m$ |
| $e$ | ... | $-C_0^2$, invariant of the ruled surface $S_m$ |
| $\delta$ | ... | $D_m^h \cdot C_0$, intersection number of $D_m^h$ and $C_0$ |

3.B. **Properties of the construction.** The following two propositions that describe features of the morphisms defined in (3.4) will be shown in Section 3.C below.

The first proposition gives a formula for the numerical class of the log canonical bundle. This is later used in Section 6 to give a relation between the logarithmic Kodaira dimension of $S \setminus D$, the genus of the base curve and the number of fibers contained in $D^v$.

**Proposition 3.5.** *There exists an effective divisor $E' \subset S$, whose support is exactly the exceptional locus of $\rho_{k_1} : S \to S_{k_1}$, such that the following equality of numerical classes holds.*

$$K_S + D \equiv (e + \delta + 2g(C) - 2)\rho_m^*(F_m) + D^v + E'.$$

We will later be interested in reducing to a situation where the horizontal components are isolated in $D$. The second proposition gives a criterion that together with Proposition 3.5 can be used to guarantee that $D^h_{k_2}$ and $D^v_{k_2}$ intersect only in a controllable manner, if at all.

**Proposition 3.6.** *Using the notation of Proposition 3.5, let $t \in C$ be a point such that the set-theoretic fiber $\operatorname{supp}(S_t)$ is not contained in the support of $D^v + E'$. Then $D^h_{k_2}$ and $D^v_{k_2}$ do not intersect over $t$, i.e., $t \notin \pi_{k_2}(D^h_{k_2} \cap D^v_{k_2})$.*

3.C. **Proofs of Propositions 3.5 and 3.6.** The proofs are not very complicated. They do, however, require some preliminary computations.

**Lemma 3.7.** *Let $t \in C$ be a point and $i < m$ a number such that $S_{i,t}$ is reducible. Then either $S_{i,t}$ contains at least two $(-1)$-curves, or it contains exactly one, but with multiplicity more than one.*

*Proof.* By blowing down vertical $(-1)$-curves disjoint from $S_{i,t}$, we can assume without loss of generality that $i = 0$, and that all vertical $(-1)$-curves blown down in Algorithm 3.3 lie over $t$. We will then prove the statement by induction on $m - i$:
*Start of induction, $i = m - 1$* In this case, $S_{m-1,t}$ contains exactly two $(-1)$ curves.
*Induction step* Suppose $i < m - 1$ and assume that the statement holds for $S_{i+1}$. Set $x := \beta_i(E_i) \subset S_{i+1}$. Then there are three possibilities:

(3.7.1) The point $x$ is contained in two vertical $(-1)$-curves. In this case, the curve $E_i$ appears in $S_{i,t}$ with multiplicity more than one.

(3.7.2) The point $x$ is contained in exactly one vertical $(-1)$-curve $E \subset S_{i+1,t}$. In this case the number of $(-1)$-curves in $S_{i,t}$ equals the number of $(-1)$-curves in $S_{i+1,t}$, and the multiplicity of $E_i$ in $S_{i,t}$ is at least the multiplicity of $E$ in $S_{i+1,t}$.

(3.7.3) The point $x$ is not contained in a vertical $(-1)$-curve. Then $S_{i,t}$ contains at least two $(-1)$ curves.

In either case, the claim is shown. This ends the proof of Lemma 3.7. □

**Corollary 3.8.** *Let $k_1 \leq j \leq m$ and $E \subseteq S_j$ a vertical $(-1)$-curve. Then $D^h_j \cdot E = 1$.*

*Proof.* Let $\beta := \beta_j \circ \beta_{j-1} \circ \cdots \circ \beta_{k_1} : S_{k_1} \to S_j$. By construction, $\beta^{-1}(E)$ contains a $(-1)$-curve, $E' \subseteq \beta^{-1}(E) \subset S_{k_1}$. By definition of $k_1$, we have $D^h_{k_1} \cap E' \neq \emptyset$. Therefore $D^h_j \cap E \supseteq \beta(D^h_{k_1} \cap E') \neq \emptyset$ and hence

$$D^h_j \cdot E \geq 1. \tag{3.8.1}$$

Note that (3.8.1) holds for *any* vertical $(-1)$-curve in $S_j$.

Let $S_{j,t}$ be the fiber containing $E$. By Assumption 3.1 $D^h_j \cdot E \leq D^h_j \cdot S_{j,t} = 2$. Assume that $D^h_j \cdot E = 2$. Then the multiplicity of $E$ in $S_{j,t}$ must be one and $D^h_j$ cannot intersect any component of $S_{j,t}$ other than $E$. But by Lemma 3.7, there has to be another vertical $(-1)$-curve $E'' \subset S_{j,t}$, which is then disjoint from $D^h_j$. This, however, contradicts (3.8.1) applied for $E''$, hence $D^h_j \cdot E < 2$ and the statement follows. □

**3.9** (*Proof of Proposition 3.5*)**.** The classical formula for the canonical bundle of a blow-up surface states:

$$K_{S_i} \equiv \beta_i^*(K_{S_{i+1}}) + E_i,$$

Next we wish to express $D^h_i$ in terms of $E_i$ and the pull-back of $D^h_{i+1}$. Depending on $i$, there are two possibilities:

$\boldsymbol{i < k_1}$**:** In this case $E_i$ and $D_i^h$ are disjoint by construction, so $D_i^h \equiv \beta_i^*(D_{i+1}^h)$ and therefore

$$K_{S_i} + D_i^h \equiv \beta_i^*(K_{S_{i+1}} + D_{i+1}^h) + E_i.$$

$\boldsymbol{i \geq k_1}$**:** In this case $D_i^h \cdot E_i = 1$ by Corollary 3.8, hence $D_i^h \equiv \beta_i^*(D_{i+1}^h) - E_i$ and therefore

$$K_{S_i} + D_i^h \equiv \beta_i^*(K_{S_{i+1}} + D_{i+1}^h).$$

In summary, we have

(3.9.1) $$K_S + D \equiv \rho^*(K_{S_m} + D_m^h) + D^v + E',$$

where $E'$ is an effective divisor supported on the exceptional locus of the morphism $\rho_{k_1} : S \to S_{k_1}$. The standard formula [Har77, V. Cor. 2.11] for the canonical bundle of a ruled surface and a simple intersection number calculation yields that

$$K_{S_m} \equiv -2C_0 + (2g(C) - 2 - e)F_m \quad \text{and} \quad D_m^h \equiv 2C_0 + (\delta + 2e)F_m.$$

Combined with (3.9.1) this finishes the proof of Proposition 3.5. □

**3.10** (*Proof of Proposition 3.6)*. Let $t \in C$ be a point as in the statement of Proposition 3.6. Assume to the contrary, i.e., that $t \in \pi_{k_2}(D_{k_2}^h \cap D_{k_2}^v)$. Observe that with this assumption the "while" condition in Step 2 of Algorithm 3.3 stopped only because there were no further $(-1)$-curves in the fiber over $t$. This implies that $S_{k_2,t}$ is reduced, irreducible and contained in $D_{k_2}^v$:

(3.10.1) $$S_{k_2,t} \simeq \mathbb{P}^1 \quad \text{and} \quad S_{k_2,t} \subseteq \operatorname{supp}\big((\rho_{k_2})_*(D^v)\big) = \operatorname{supp}\big((\rho_{k_2})_*(D^v + E')\big).$$

In contrast to (3.10.1), since $E'$ is supported exactly on the exceptional locus of $\rho_{k_1}$, the assumption of Proposition 3.6 says precisely that

(3.10.2) $$\operatorname{supp}(S_{k_1,t}) \not\subset \operatorname{supp}\big((\rho_{k_1})_*(D^v)\big) = \operatorname{supp}\big((\rho_{k_1})_*(D^v + E')\big).$$

Now let

(3.10.3) $$j := \max\big\{i \mid \operatorname{supp}(S_{i,t}) \not\subset \operatorname{supp}\big((\rho_i)_*(D^v + E')\big)\big\}.$$

It follows by (3.10.1) and (3.10.2) that

(3.10.4) $$k_1 \leq j < k_2.$$

Loosely speaking, the exceptional curve $E_j$ is the last $(-1)$-curve contracted over $t$ that is not in the image of $D^v + E'$. The following two statements follow immediately from the choice of $j$.

(3.10.5) $E_j$ is contained in the fiber over $t$, i.e., $E_j \subset S_{j,t}$
(3.10.6) $E_j$ is not contained in the image of $D^v + E'$, i.e., $E_j \not\subset \operatorname{supp}\big((\rho_j)_*(D^v)\big)$.

The choice of $j$ and the "if" statement in Step 2 of Algorithm 3.3 guarantee that $E_j$ is the only $(-1)$-curve contained in $S_{j,t}$. In that case Lemma 3.7 asserts that the multiplicity of $E_j$ in $S_{j,t}$ is at least 2. In addition, the first inequality of (3.10.4) and Corollary 3.8 assert that $D_j^h$ intersects $E_j$ non-trivially. Then by Assumption 3.1 $D_j^h$ does not intersect any component of the fiber $S_{j,t}$ other than $E_j$. Then (3.10.5) and (3.10.6) imply that

$$D_j^h \cap D_j^v \cap S_{j,t} = \emptyset.$$

This, combined with the second inequality in (3.10.4) above contradicts the choice of $k_2$ as the index marking the end of Step 2 of Algorithm 3.3.

# PART 2. **PROOF OF THE MAIN THEOREM**

## 4. Setup of notation

In this section, we briefly fix notation used throughout the proof of Theorem 1.4. The proof will be given in Sections 5–7 for the cases when the logarithmic Kodaira dimension of $S^\circ$ is 1, 0 or $-\infty$, respectively. As one might expect, the case of $\kappa(S^\circ) = 0$ is by far the longest and most involved.

*Notation* 4.1. Throughout the rest of the article, we keep the notation and assumptions of Theorem 1.4. We fix a smooth projective compactification $S$ of $S^\circ$ such that $D = S \setminus S^\circ \subset S$ is a simple normal crossing divisor. Furthermore, let $X$ be a smooth projective variety and $f : X \to S$ a morphism such that $X \setminus f^{-1}(D) \simeq X^\circ$ and $f|_{X^\circ} = f^\circ$.

Part of the argumentation involves the log minimal model of $(S, D)$. We will therefore adhere to the notation introduced in section 2.A. In particular, we use

$$\phi : (S, D) \to (S_\lambda, D_\lambda)$$

to denote the birational morphism from $S$ to its logarithmic minimal model that is described in Fact 2.1.

## 5. Logarithmic Kodaira dimension 1

If $\kappa(S^\circ) = 1$, the statement of Theorem 1.4 follows almost immediately from the logarithmic minimal model program.

*Proof of Theorem 1.4 when $\kappa(S^\circ) = 1$.* By Fact 2.1, we can run the logarithmic minimal model program and find a birational morphism $\phi : S \to S_\lambda$ from $S$ to a normal surface $S_\lambda$ such that the associated log-canonical divisor $K_{S_\lambda} + D_\lambda$ on $S_\lambda$ is nef.

The logarithmic abundance theorem in dimension 2, Fact 2.4, then asserts that for $n \gg 0$ the linear system $|n(K_{S_\lambda} + D_\lambda)|$ yields a morphism to a curve $\pi_\lambda : S_\lambda \to C$, such that $K_{S_\lambda} + D_\lambda$ is trivial on the general fiber $F_\lambda$ of $\pi_\lambda$. Likewise, if $\pi := \pi_\lambda \circ \phi$, and $F \subset S$ is a general fiber of $\pi$, then $K_S + D$ is trivial on $F$. It follows that $F$ is either an elliptic curve that does not intersect $D$, or that $F$ is a rational curve that intersects $D$ in two points. It follows, in the former case from [Kov96, Thm. 1, Cor. 3.2] and in the latter case from [Kov00, 0.2], that $f^\circ$ is isotrivial over $F \setminus D$, and therefore $\mathrm{Var}(f^\circ) \le 1 = \kappa(S^\circ)$. □

## 6. Logarithmic Kodaira dimension 0

Throughout the present section, we maintain the notation and assumptions of Theorem 1.4 and Section 4 and assume that $\kappa(S^\circ) = 0$.

As the proof is rather long, we subdivide it into several steps. We start in Section 6.A by recalling a result of Viehweg and Zuo on which much of the argumentation is based. As a first application, we will in Section 6.B reduce to the situation where $S$ is uniruled. In Section 6.C we will further reduce to the case where $S$ is birationally ruled over a curve.

This makes it possible in Section 6.D to employ the results of Chapter 3 to construct a birational model of $S$ to which the aforementioned result of Viehweg and Zuo can be applied. The application itself, carried out in Sections 6.E–6.F, shows that $\mathrm{Var}(f^\circ) = 0$ and finishes the proof of Theorem 1.4.

### 6.A. **A result of Viehweg and Zuo.**

The argumentation relies on the following result describing the sheaf of logarithmic differentials on the base of a family of canonically polarized varieties. Note that we are still using Notation 4.1.

**Theorem 6.1.** [VZ02, Thm. 1.4(i)]. *There exists an integer $n > 0$ and an invertible subsheaf $\mathscr{A} \subset \mathrm{Sym}^n\, \Omega^1_S(\log D)$ of Kodaira dimension $\kappa(\mathscr{A}) \ge \mathrm{Var}(f^\circ)$.* □

We will show that $\operatorname{Var}(f^\circ) = 0$ by a detailed analysis of $\Omega^1_S(\log D)$. Essentially, we prove that for all numbers $n$ and locally free subsheaves $\mathscr{A} \subset \operatorname{Sym}^n \Omega^1_S(\log D)$, the Kodaira dimension of $\mathscr{A}$ is never positive, $\kappa(\mathscr{A}) \leq 0$.

6.B. **Reduction to the uniruled case.** A surface $S$ with $\kappa(S^\circ) = 0$, of course, need not be uniruled. Using the result of Viehweg and Zuo, however, we will show that any family of canonically polarized varieties over a non-uniruled surface $S$ with $\kappa(S^\circ) = 0$ is isotrivial.

**Proposition 6.2.** *If $S$ is not uniruled, i.e., if $\kappa(S) \geq 0$, then $\operatorname{Var}(f^\circ) = 0$.*

We prove Proposition 6.2 using two lemmata.

**Lemma 6.3.** *If $n \in \mathbb{N}$ is sufficiently large and divisible, then*

$$\mathscr{O}_{S_\lambda}(n(K_{S_\lambda} + D_\lambda)) = \mathscr{O}_{S_\lambda}. \tag{6.3.1}$$

*In particular, the log canonical $\mathbb{Q}$-divisor $K_{S_\lambda} + D_\lambda$ is numerically trivial.*

*Proof.* (6.3.1) is an immediate consequence of the assumption $\kappa(S^\circ) = 0$ and the logarithmic abundance theorem in dimension 2, Fact 2.4, which asserts that the linear system $|n(K_{S_\lambda} + D_\lambda)|$ is basepoint-free. □

**Lemma 6.4.** *If $\kappa(S) \geq 0$, then $S_\lambda$ is $\mathbb{Q}$-Gorenstein, $K_{S_\lambda}$ is numerically trivial and $D_\lambda = \emptyset$.*

*Proof.* Lemma 6.3 together with the assumption that $|nK_S| \neq \emptyset$ for large $n$ imply that $\phi$ contracts all irreducible components of $D$, and all divisors in any linear system $|nK_S|$, for all $n \in \mathbb{N}$. Hence the claim follows. □

*Proof of Proposition 6.2.* We argue by contradiction and assume to the contrary that both $\kappa(S) \geq 0$ and $\operatorname{Var}(f^\circ) \geq 1$. Let $H \in \operatorname{Pic}(S_\lambda)$ be an arbitrary ample line bundle.

*Claim* 6.4.1. The reflexive sheaf of differentials $(\Omega^1_{S_\lambda})^{\vee\vee}$ has slope $\mu_H\big((\Omega^1_{S_\lambda})^{\vee\vee}\big) = 0$, but it is not semistable with respect to $H$.

*Proof of Claim 6.4.1.* Fix a sufficiently large number $m > 0$ and a general curve $C_\lambda \in |mH|$. Flenner's variant of the Mehta-Ramanathan theorem, [Fle84, Thm. 1.2], then ensures that if $(\Omega^1_{S_\lambda})^{\vee\vee}$ is semistable, then so is its restriction $(\Omega^1_{S_\lambda})^{\vee\vee}|_{C_\lambda}$.

By the general choice, $C_\lambda$ is contained in the smooth locus of $S_\lambda$ and stays off the fundamental points of $\phi^{-1}$. The birational morphism $\phi$ will thus be well-defined and isomorphic along $C := \phi^{-1}(C_\lambda)$. Lemma 6.4 then asserts that

$$\mu_H\big((\Omega^1_{S_\lambda})^{\vee\vee}\big) = \frac{K_{S_\lambda} \cdot C_\lambda}{2m} = 0,$$

which shows the first claim.

Lemma 6.4 further implies that $\operatorname{codim}_{S_\lambda} \phi(D) \geq 2$, and so $C$ is disjoint from $D$. The unstability of $(\Omega^1_{S_\lambda})^{\vee\vee}$ can therefore be checked using the identifications

$$(\Omega^1_{S_\lambda})^{\vee\vee}|_{C_\lambda} \cong \Omega^1_{S_\lambda}\big|_{C_\lambda} \cong \Omega^1_S\big|_C \cong \Omega^1_S(\log D)|_C. \tag{6.4.2}$$

Since symmetric powers of semistable vector bundles over curves are again semistable [HL97, Cor. 3.2.10], in order to prove Claim 6.4.1, it suffices to show that there exists a number $n \in \mathbb{N}$ such that $\operatorname{Sym}^n \Omega^1_S(\log D)|_C$ is not semistable. For that, use the identifications (6.4.2) to compute

$$\begin{aligned}
\deg_C \operatorname{Sym}^n \Omega^1_S(\log D)|_C &= \mathit{const}^+ \cdot \deg_C \Omega^1_S\big|_C \\
&= \mathit{const}^+ \cdot \deg_{C_\lambda} (\Omega^1_{S_\lambda})^{\vee\vee}|_{C_\lambda} && \text{(6.4.2)} \\
&= \mathit{const}^+ \cdot (K_{S_\lambda} \cdot C_\lambda) = 0. && \text{Lemma 6.4}
\end{aligned}$$

Hence, to prove unstability it suffices to show that $\operatorname{Sym}^n \Omega^1_S(\log D)|_C$ contains a subsheaf of positive degree.

Theorem 6.1 implies that there exists an integer $n > 0$ such that $\operatorname{Sym}^n \Omega^1_S(\log D)$ contains an invertible subsheaf $\mathscr{A}$ of Kodaira dimension $\kappa(\mathscr{A}) \geq 1$. But by general choice of $C_\lambda$, this in turn implies that $\deg_C(\mathscr{A}|_C) > 0$, which shows the required unstability. This ends the proof of Claim 6.4.1. □

Claim 6.4.1 implies that $\Omega^1_{S_\lambda}|_{C_\lambda}$ has a subsheaf of positive degree or, equivalently, that it has a quotient of negative degree. On the other hand, Miyaoka's criterion for uniruledness, [Miy87, Cor. 8.6] or [KST05], asserts that then $S$ is uniruled, leading to a contradiction. □

In view of Proposition 6.2, it suffices to prove Theorem 1.4 under the following additional assumption that we maintain for the rest of Section 6.

**Assumption 6.5.** *In addition to the notation and assumptions introduced above we further assume that $S$ is uniruled.*

6.C. **Reduction to birationally ruled surfaces.** We will now show that $S^\circ$ is dominated by curves that are images of $\mathbb{A}^1 \setminus \{0\}$. We will then, in Proposition 6.8, conclude that unless $f^\circ$ is isotrivial, a general point of $S^\circ$ is contained in exactly one image of $\mathbb{A}^1 \setminus \{0\}$. This will exhibit $S$ as a birationally ruled surface.

**Proposition 6.6.** *The surface $S$ is dominated by a family of rational curves that intersect $D$ in two points, but it is not dominated by rational curves intersecting $D$ in one point.*

*Remark* 6.7. In Proposition 6.6, the number of intersection points is to be understood in the sense of Definition 2.5.

*Proof of Proposition 6.6.* Recall from [KMcK99, Thm. 1.1] that $S$ is dominated by rational curves that intersect $D$ in one point iff $\kappa(S^\circ) = -\infty$, which is not the case.

*Claim* 6.7.1. The smooth locus $S_\lambda \setminus \operatorname{Sing}(S_\lambda)$ is dominated by rational curves intersecting $D_\lambda$ in two points.

*Proof of Claim 6.7.1.* We aim to apply [KMcK99, Prop. 1.4(3)], and so we need that
- the log canonical divisor $K_{S_\lambda} + D_\lambda$ is numerically trivial, and that
- the boundary divisor $D_\lambda$ is not empty.

The numerical triviality of $K_{S_\lambda} + D_\lambda$ was shown in Lemma 6.3 above. To show that $D_\lambda \neq \emptyset$, we argue by contradiction, and assume that $D_\lambda = \emptyset$. Set

$$S^1_\lambda := S_\lambda \setminus \underbrace{\phi(\text{exceptional set of } \phi)}_{\text{finite, contains } \phi(D)}.$$

Then $S^1_\lambda$ is the complement of a finite set and $\phi^{-1}|_{S^1_\lambda}$ is a well-defined open immersion. Let $f_\lambda := \phi \circ f$. Then $X|_{f_\lambda^{-1}(S^1_\lambda)} \to S^1_\lambda$ is a smooth family of canonically polarized varieties. Consider the following diagram:

$$\begin{tikzcd}
X \arrow[d, "f_\lambda"'] & & \tilde{X} := X \times_{S_\lambda} \tilde{S} \arrow[ll] \arrow[d, "\tilde{f}"] \\
S_\lambda & \tilde{S}_\lambda \arrow[l, "\alpha", "\text{index-one-cover}"'] & \tilde{S} \arrow[l, "\beta", "\text{log resolution}"']
\end{tikzcd}$$

where $\alpha$ is the index-one-cover described in [KM98, 5.19], and $\beta$ is the minimal desingularization of $\tilde{S}_\lambda$ composed with blow-ups of smooth points so that $\beta^{-1}(\tilde{S}_\lambda \setminus \alpha^{-1}(S^1_\lambda))$ is a divisor with at most simple normal crossings.

By Lemma 6.3, $K_{S_\lambda}$ is torsion. Since $\alpha$ is étale in codimension one this implies that $K_{\tilde{S}_\lambda}$ is trivial. Furthermore, $\tilde{S}_\lambda$ has only canonical singularities: we have already noted in Remark 2.2 that the singularities of $S_\lambda$ are log-terminal, i.e., they have minimal discrepancy $> -1$. Then by [KM98, Prop. 5.20] the minimal discrepancy

of the singularities of $\tilde{S}_\lambda$ is also $> -1$, and as $K_{\tilde{S}_\lambda}$ is Cartier, the discrepancies actually must be integral and hence $\geq 0$, cf. [KM98, proof of Cor. 5.21]. Consequently,

$$K_{\tilde{S}} = \underbrace{\beta^*(K_{\tilde{S}_\lambda})}_{\cong \mathcal{O}_{\tilde{S}}} + (\text{effective and } \beta\text{-exceptional}). \tag{6.7.2}$$

This in turn has two further consequences:

i) $\kappa(K_{\tilde{S}}) = 0$. In particular, $\tilde{S}$ is not uniruled.

ii) If we set $\tilde{S}^1 := (\alpha \circ \beta)^{-1}(S^1_\lambda)$ then $\tilde{X}|_{\tilde{f}^{-1}(\tilde{S}_1)} \to \tilde{S}^1$ is again a smooth family of canonically polarized varieties. Letting $\tilde{D} := \tilde{S} \setminus \tilde{S}^1$ then $\tilde{D}$ is exactly the $\beta$-exceptional set, and (6.7.2) implies that
$$\kappa(\tilde{S}^1) = \kappa(\underbrace{K_{\tilde{S}} + \tilde{D}}_{\text{effective, } \beta\text{-exceptional}}) = 0.$$

In particular, Proposition 6.2 applies to $\tilde{f} : \tilde{X} \to \tilde{S}$ and shows that $\tilde{S}$ is uniruled. This is a contradiction and thus the proof of Claim 6.7.1 is complete. □

If $\phi(D) \subset D_\lambda \cup \mathrm{Sing}(S_\lambda)$, i.e., if all connected components of $D$ are either mapped to singular points, or to divisors, Claim 6.7.1 and Remark 2.6 immediately imply Proposition 6.6. Likewise, if $S_\lambda$ were smooth, Proposition 2.8 on small set avoidance would imply that almost all curves in the family stay off the isolated zero-dimensional components of $\phi(D)$, and Proposition 6.6 would again hold. In the general case, when $S_\lambda$ is singular, and $d_1, \ldots, d_r$ are smooth points of $S_\lambda$ that appear as connected components of $\phi(D)$, a little more care is required.

If $D'$ is the union of connected components of $D$ which are contracted to the set of points $\{d_1, \ldots, d_r\} \subset S_\lambda$, it is clear that the birational morphism $\phi : S \to S_\lambda$ factors via the contraction of $D'$, i.e. there exists a diagram

$$\begin{tikzcd} S \arrow[r, "\alpha"'] \arrow[rr, bend left, "\phi"] & S' \arrow[r, "\beta"'] & S_\lambda \end{tikzcd}$$

where $S'$ is smooth, and $\alpha$ maps the connected components of $D'$ to points $d'_1, \ldots, d'_r \in S'$ and is isomorphic outside of $D'$.

Now, if $D'' := D \setminus D'$, the above argument shows that $S'$ is dominated by rational curves that intersect $\alpha(D'')$ in two points. Since $S'$ is smooth, Proposition 2.8 on small set avoidance applies and shows that almost all of these curves do not contain any of the $d'_i$. Therefore, we have seen that most of the curves in question intersect $\alpha(D)$ in two points. Remark 2.6 then completes the proof. □

**Proposition 6.8.** *Either* $\mathrm{Var}(f^\circ) = 0$*, or there exists a smooth curve* $C$ *and morphisms*

$$\begin{tikzcd} C & \tilde{S} \arrow[l, "\pi"', "\text{birat. ruling}"] \arrow[r, "\psi", "\text{birational}"'] & S \end{tikzcd}$$

*such that*

(6.8.1) *$\tilde{S}$ is a smooth surface and $\tilde{D} := \psi^{-1}(D)$ is a divisor with simple normal crossing support.*

(6.8.2) *If $t \in C$ is a general point, then the fiber $\tilde{S}_t := \pi^{-1}(t)$ is isomorphic to $\mathbb{P}^1$, and intersects $\tilde{D}$ in exactly two points. In particular, $\tilde{D} \cdot \tilde{S}_t = 2$ for all $t \in C$.*

(6.8.3) *The restriction of $f^\circ$ to any fiber of $\pi \circ \psi^{-1}|_{S^\circ}$ is isotrivial.*

(6.8.4) *The morphism $\psi$ is birational, and isomorphic over $S^\circ$. In particular, $\pi$ induces a fibration $\pi \circ \psi^{-1}|_{S^\circ} : S^\circ \to C$.*

*Proof.* If $\mathrm{Var}(f^\circ) = 0$, there is nothing to prove, so we may assume that $\mathrm{Var}(f^\circ) > 0$.

By Proposition 6.6, there exists a proper curve $C' \subset \mathrm{Chow}(S)$ such that general points $t \in C'$ are associated with irreducible, reduced rational curves $\ell_t$ intersecting $D$ in exactly

two points, in the sense of Definition 2.5. Then by [Kov00, Thm. 0.2] the restriction of the family $f$ to a general curve $(\ell_t)_{t\in C'}$ is isotrivial, so (6.8.3) follows from the rest of the statement.

Let $\tilde{S}'$ be the restriction of the universal family over $\mathrm{Chow}(S)$ to $C'$ and $\psi' : \tilde{S}' \to S$ the restriction of the cycle morphism. Finally, let $C$ be the normalization of $C'$, $\tilde{S}$ the normalization of $\tilde{S}' \times_{C'} C$ and $\psi : \tilde{S} \to S$ the morphism induced by $\psi'$. After blowing up further, we may assume that $\tilde{S}$ is smooth and that $\tilde{D} := (\psi^{-1}(D))_{\mathrm{red}}$ has only simple normal crossings.

(6.8.1) and (6.8.2) hold by construction. The last part of (6.8.2) follows from the fact that for a general $t \in C$, $\tilde{D}$ intersects $\tilde{S}_t$ transversally and the numerical class of $\tilde{S}_t$ is independent of $t$. By Zariski's main theorem [Har77, Thm. V.5.2], the proof of Proposition 6.8 is finished if we show that $\psi$ is birational, and that it is finite over $S^\circ = S \setminus D$.

Birationality. Since we are working in characteristic $0$, it suffices to show that $\psi$ is generically injective. Assume to the contrary that a general point in $S^\circ$ is contained in more than one of the $\ell_t$'s.

Fix a general $t \in C'$. Then the associated curve $\ell_t$ intersects $D$ in exactly two points. Further, there exists an open set $\ell_t^\circ \subset \ell_t$ such that any $x \in \ell_t^\circ$ satisfies the following:

- $x$ is a general point of $S^\circ$, and
- there exists a point $t_x \in C'$ such that the associated curve $\ell_{t_x}$ contains $x$, is different from $\ell_t$, and intersects $D$ in exactly two points,

Since the $\ell_{t_x}$ dominate $S$, and since $f$ is isotrivial both over $\ell_t$ and over any of the $\ell_{t_x}$, $f$ must be isotrivial, contrary to our assumption.

Finiteness. If there was a point $s \in S^\circ$ that was contained in infinitely many of the curves $(\ell_t)_{t\in C'}$, then the isotriviality of the restrictions $f|_{\ell_t}$ would again imply that $f$ is isotrivial over $S$, contradicting our assumptions. □

To prove Theorem 1.4, we may replace $S$ by $\tilde{S}$ and $X$ by a desingularization of $X \times_S \tilde{S}$. We will thus make the following additional assumption that we will maintain without loss of generality for the rest of the present section.

**Assumption 6.9.** *Assume that there exists a morphism $\pi : S \to C$ to a smooth curve $C$, with the following property: If $t \in C$ is a general point, then the fiber $S_t := \pi^{-1}(t)$ is isomorphic to $\mathbb{P}^1$, and intersects $D$ in exactly two points. In particular, $D \cdot S_t = 2$ for all $t \in C$.*

6.D. **Construction of a good model of $S$.** In order to apply Theorem 6.1 to our setup, we need to study the restriction of the sheaf of logarithmic differentials to components of the boundary. While the restriction to isolated components is easily described using Lemma 2.14, in general we have very little control over the intersection graph of the boundary divisor. It seems therefore rather difficult to describe logarithmic differentials directly in this naïve manner.

To overcome this difficulty and to simplify the intersection graph, we recall the results of Section 3 and apply Algorithm 3.3 to the birationally ruled surface $\pi : S \to C$. We have seen in Proposition 3.6 that in the intermediate surface $S_{k_2}$, the horizontal components of the boundary divisor $D^h_{k_2}$ are disjoint from the vertical components $D^v_{k_2}$ as long as $D^v + E'$ does not contain an entire fiber of $\pi$. This makes the analysis of the sheaf of logarithmic differentials much easier. Unfortunately, the image $\rho_{k_2}(D)$ need not be a normal crossing divisor. We construct a log resolution of $\big(S_{k_2}, \rho_{k_2}(D)\big)$ as follows.

*Construction* 6.10. Let $S^\circ_{k_2} \subset S^\circ$ be the maximal open subset where $\rho_{k_2}$ is isomorphic. The difference $S^\circ \setminus S^\circ_{k_2}$ is then contained in finitely many fibers. By definition, we can view $S^\circ_{k_2}$ also as an open subset of $S_{k_2}$, and observe that

$$S_{k_2} \setminus S^\circ_{k_2} = D^h_{k_2} \cup D^v_{k_2} \cup \{\text{finitely many isolated points}\}.$$

Let $\beta : S_\mu \to S_{k_2}$ be the minimal log resolution of the pair $(S_{k_2}, S_{k_2} \setminus S^\circ_{k_2})$. If $X_\mu$ is a desingularization of the pull-back $X \times_{S_{k_2}} S_\mu$, we obtain a diagram as follows:

$$\begin{tikzcd}
X_\mu \arrow[rr, "f_\mu"] \arrow[d] & & S_\mu \arrow[d, "\beta"'] \arrow[drr, bend left, "\pi_\mu = \pi_{k_2} \circ \beta"] & & \\
X \arrow[r, "f"'] & S \arrow[r, "\rho_{k_2}"'] & S_{k_2} \arrow[r, "\gamma"'] & S_m \arrow[r, "\pi_m"'] & C
\end{tikzcd}
\qquad \text{with} \qquad
\begin{array}{ll}
\pi & = \pi_m \circ \gamma \circ \rho_{k_2} \\
\pi_{k_2} & = \pi_m \circ \gamma \\
\pi_\mu & = \pi_m \circ \gamma \circ \beta
\end{array}$$

Again, the rational map $\beta^{-1} \circ \rho_{k_2}$ is an isomorphism over $S^\circ_{k_2}$, so that we can view $S^\circ_{k_2}$ as a subset of $S_\mu$. The morphism $f_\mu$ is smooth over $S^\circ_{k_2} \subset S_\mu$, and to show that $f^\circ$ is isotrivial, it suffices to prove isotriviality for $f_\mu$. Finally, let $D_\mu := S_\mu \setminus S^\circ_{k_2}$. Then $D_\mu$ is a simple normal crossing divisor that we decompose into horizontal and vertical components, $D_\mu = D^h_\mu \cup D^v_\mu$, as before. Note that by Assumption 6.9 $D^h_\mu$ is a double section, in particular, it has at most two irreducible components.

*Notation* 6.11. We have applied Algorithm 3.3 to the birationally ruled surface $\pi : S \to C$ in order to construct $S_{k_2}$ and $S_m$. Throughout the remainder of the present Section 6, we maintain Notation 3.4 that was introduced on page 8 along with Algorithm 3.3.

In particular, we let $C_0 \subset S_m$ be the distinguished section of $\pi_m$ with the minimal self-intersection number, $e = -C_0^2$ and $\delta = D^h_m \cdot C_0$.

6.E. **Another application of Theorem 6.1.** Fix an irreducible component $D^{h,1}_\mu \subset D^h_\mu$. Using Proposition 3.6 we will be able to show in Section 6.F below that $D^{h,1}_\mu$ is either rational or elliptic, and compute an upper bound for the number of intersection points between $D^{h,1}_\mu$ and other components of $D_\mu$. Theorem 6.1 will then apply to $S_\mu$ and yield the following proposition.

**Proposition 6.12.** *Let $D^{h,1}_\mu \subset D^h_\mu$ be an irreducible component. If either one of the following holds:*

(6.12.1) $D^{h,1}_\mu$ *is elliptic and isolated in $D_\mu$, or*

(6.12.2) $D^{h,1}_\mu$ *is rational and intersects other components of $D_\mu$ in at most two points,*

*then* $\mathrm{Var}(f^\circ) = 0$.

*Proof.* We argue by contradiction and assume $\mathrm{Var}(f^\circ) > 0$. By Theorem 6.1 there exists a number $n > 0$ and an invertible subsheaf $\mathscr{A}_\mu \subset \mathrm{Sym}^n \Omega^1_{S_\mu}(\log D_\mu)$ of Kodaira dimension $\kappa(\mathscr{A}_\mu) > 0$.

If $F_\mu \subset S_\mu$ is a general fiber of $\pi_\mu$, then $F_\mu$ is isomorphic to $\mathbb{P}^1$ and intersects $D_\mu$ transversally in exactly two points. Then the logarithmic normal bundle sequence (2.14.1) from Lemma 2.14 is split. The restriction $\Omega^1_{S_\mu}(\log D_\mu)|_{F_\mu}$ is therefore trivial, and so is $\mathrm{Sym}^n \Omega^1_{S_\mu}(\log D_\mu)|_{F_\mu}$. It follows that the restriction of $\mathscr{A}_\mu$ to $F_\mu$ is a trivial subsheaf of $\mathrm{Sym}^n \Omega^1_{S_\mu}(\log D_\mu)|_{F_\mu}$. This has two consequences. First, the restriction of $\mathscr{A}_\mu$ to $D^{h,1}_\mu$ must have positive Kodaira dimension. Second, the natural map between restrictions, $\mathscr{A}_\mu|_{D^{h,1}_\mu} \to \mathrm{Sym}^n \Omega^1_{S_\mu}(\log D_\mu)|_{D^{h,1}_\mu}$, is not zero.

On the other hand, sequence (2.14.2) from Lemma 2.14 gives

$$0 \to \underbrace{\Omega^1_{D^{h,1}_\mu}(\log(D_\mu - D^{h,1}_\mu)|_{D^{h,1}_\mu})}_{=:\mathscr{L}} \to \underbrace{\Omega^1_{S_\mu}(\log D_\mu)|_{D^{h,1}_\mu}}_{=:\mathscr{R}} \to \mathcal{O}_{D^{h,1}_\mu} \to 0,$$

where $\mathscr{L} \in \mathrm{Pic}(D^{h,1}_\mu)$ is a line bundle of degree

$$\deg \mathscr{L} = 2g(D^{h,1}_\mu) - 2 + \#\{\text{intersection points of } D^{h,1}_\mu \text{ with other components of } D_\mu\}.$$

If $D^{h,1}_\mu$ is elliptic and isolated in $D_\mu$, then $\deg \mathscr{L} = 0$. Then $\mathscr{R}$ is semistable of degree 0, and so is $\mathrm{Sym}^n \mathscr{R}$. Likewise, if $D^{h,1}_\mu$ is rational and intersects other components of

$D_\mu$ in at most two points, then $-2 \le \deg \mathscr{L} \le 0$. Then $\mathscr{R}$ is a sum of line bundles of non-positive degree, and so is $\mathrm{Sym}^n \mathscr{R}$. In both cases, we have $\deg(\mathscr{A}_\mu|_{D_\mu^{h,1}}) \le 0$. A contradiction. □

6.F. **Computation of genera and intersection points.** In order to apply Proposition 6.12, we need to compute the genus of $D_\mu^{h,1}$ and the number of intersection points between $D_\mu^{h,1}$ and other components of $D_\mu$. While it is possible to write down a (complicated) formula that involves both pieces of information, we found it easier to consider the cases where $D_\mu^h$ is reducible, respectively irreducible, separately in Sections 6.F.1 and 6.F.2.

The following simple observation helps to count the number of intersection points in either case.

*Observation* 6.13. If $x \in D_\mu^{h,1}$ is a point of intersection between $D_\mu^{h,1}$ and other components of $D_\mu$, then, using the notation introduced in (6.10), one of the following holds:

(6.13.1) $\beta(x)$ is a singular point of $D_{k_2}^{h,1} = \beta_*(D_\mu^{h,1})$. In particular, $(\gamma \circ \beta)(x)$ is a singular point of $D_m^{h,1} = (\gamma \circ \beta)_*(D_\mu^{h,1})$.

(6.13.2) $\beta(x) \in D_{k_2}^h \cap D_{k_2}^v$. By Proposition 3.6, $\pi_\mu(x)$ is a point whose set-theoretic fiber $\mathrm{supp}(S_{\pi_\mu(x)})$ is contained in the support of $D^v + E'$.

(6.13.3) $D_{k_2}^h$ is reducible and $\beta(x) \in D_{k_2}^{h,1} \cap D_{k_2}^{h,2}$. In particular, $D_m^h$ is reducible and $(\gamma \circ \beta)(x) \in D_m^{h,1} \cap D_m^{h,2}$. □

Formulated in more technical terms, Observation 6.13 gives the following.

**Corollary 6.14.** *If we denote the the number of points as follows,*

$$\begin{aligned} I &:= \#\{\textit{intersection points between } D_\mu^{h,1} \textit{ and other components of } D_\mu\} \\ I_1 &:= \#\{x \in D_\mu^{h,1} \mid (\gamma \circ \beta)(x) \textit{ is a singular point of } D_m^{h,1}\} \\ I_2 &:= \#\{x \in D_\mu^{h,1} \mid \mathrm{supp}(S_{\pi_\mu(x)}) \subset \mathrm{supp}(D^v + E')\} \end{aligned}$$

*then*

$$I \le I_1 + I_2 + \begin{cases} D_m^{h,1} \cdot D_m^{h,2} & \textit{if } D_m^h \textit{ is reducible} \\ 0 & \textit{otherwise} \end{cases}$$

□

It remains to compute the numbers $I_1$, $I_2$ and $D_m^{h,1} \cdot D_m^{h,2}$ in all relevant cases. Before we do that, we remark that the results of Section 3 immediately give an upper bound for the number $I_2$. For this, we maintain the notation of Section 3. In particular, we use the numbers $e$ and $\delta$ that were introduced in Notation 3.4.

**Lemma 6.15.** *If $d$ is the degree of the finite morphism $D_\mu^{h,1} \to C$, then*

$$0 \le I_2 \le -d \cdot (e + \delta + 2g(C) - 2).$$

*Proof.* It is clear from the definition that

$$I_2 \le d \cdot \#\{t \in C \mid \mathrm{supp}(S_t) \subset \mathrm{supp}(D^v + E')\}$$

On the other hand, since $\kappa(S^\circ) = 0$, it follows from Proposition 3.5 that

$$0 \le \#\{t \in C \mid \mathrm{supp}(S_t) \subset \mathrm{supp}(D^v + E')\} \le -(e + \delta + 2g(C) - 2).$$

This shows the claim. □

6.F.1. *Computation of genera and intersection points if $D^h_\mu$ is reducible.* Let $D^{h,1}_m$ and $D^{h,2}_m$ be the irreducible components of $D^h_m$ and write $D^{h,i}_m \equiv C_0 + b_i F_m$. Since $\delta = D^h_m \cdot C_0$, a simple computation shows that $D^h_m \equiv 2C_0 + (\delta + 2e)F_m$. In particular, $b_1 + b_2 = \delta + 2e$, and then the intersection number between the components is

$$D^{h,1}_m \cdot D^{h,2}_m = (C_0 + b_1 F_m) \cdot (C_0 + b_2 F_m) = -e + b_1 + b_2 = e + \delta. \tag{6.16}$$

The number $I$ can be then bounded as follows:

$$\begin{aligned} 0 \le I &\le I_1 + I_2 + D^{h,1}_m \cdot D^{h,2}_m && \text{Corollary 6.14} \\ &= I_2 + D^{h,1}_m \cdot D^{h,2}_m && \text{because } D^{h,1}_m \cong C \text{ is smooth} \\ &\le -(e + \delta + 2g(C) - 2) + D^{h,1}_m \cdot D^{h,2}_m && \text{Lemma 6.15} \\ &= 2 - 2g(C). && \text{Equation (6.16)} \end{aligned}$$

In particular, either $D^{h,1}_\mu$ is elliptic and $I = 0$, or it is rational and $I \le 2$. The prerequisites of Proposition 6.12 are thus fulfilled in any case. It follows that $\operatorname{Var}(f^\circ) = 0$, and Theorem 1.4 is shown in this case. □

6.F.2. *Computation of genera and intersection points if $D^h_\mu$ is irreducible.* Since $D^h_m \equiv 2C_0 + (\delta + 2e)F_m$, the standard formula [Har77, V. Cor. 2.11] for the numerical class of the canonical bundle of a ruled surface gives

$$K_{S_m} + D^h_m \equiv (e + \delta + 2g(C) - 2)F_m.$$

The formula [Har77, V. Ex. 1.3a] for the arithmetic genus of $D^h_m$ then says that

$$p_a(D^h_m) = \frac{(K_{S_m} + D^h_m) \cdot D^h_m}{2} + 1 = \underbrace{(e + \delta + 2g(C) - 2)}_{\le 0 \text{ by Lemma 6.15}} + 1 \le 1 \tag{6.17}$$

In particular, $D^h_\mu$ is either elliptic or rational. We treat these cases separately.

If $D^h_\mu$ is elliptic, then $g(D^h_m) = p_a(D^h_m) = 1$ and so $D^h_m$ is smooth, $I_1 = 0$. Corollary 6.14 and Equation (6.17) assert

$$I \le I_2 \le -2 \cdot (e + \delta + 2g(C) - 2) = -2 \cdot \big(p_a(D^h_m) - 1\big) = 0.$$

The elliptic curve $D^h_\mu$ is thus isolated in $D_\mu$, Proposition 6.12 implies that $\operatorname{Var}(f^\circ) = 0$, and Theorem 1.4 is shown in this case.

Therefore we may assume that $D^h_\mu$ is rational. If $D^h_m$ is singular, its arithmetic genus is exactly one and [Har77, IV. Ex. 1.8a] asserts that there are at most two points in $D^h_\mu$ that map to the singularities. Hence, whether $D^h_m$ is singular or not, $I_1$ is always bounded as $I_1 \le 2 \cdot p_a(D^h_m)$. Then Corollary 6.14, Lemma 6.15 and Equation (6.17) assert that

$$I \le I_1 + I_2 \le 2 \cdot p_a(D^h_m) - 2 \cdot (e + \delta + 2g(C) - 2) = 2.$$

Again, Proposition 6.12 applies and Theorem 1.4 is shown. □

## 7. Logarithmic Kodaira dimension $-\infty$

We maintain the notation and assumptions of Theorem 1.4 and Section 4 and assume that $\kappa(S^\circ) = -\infty$. In this case, the statement follows quickly from the logarithmic abundance result of Keel-McKernan:

**Theorem 7.1.** [KMcK99, Thm. 1.1]. *Let $S$ be a smooth projective surface and $D \subset S$ a reduced divisor with simple normal crossings. Assume that*

$$\kappa(S \setminus D) = -\infty.$$

*Then $S \setminus D$ is dominated by a family of curves that are isomorphic to $\mathbb{A}^1$.* □

*Proof of Theorem 1.4 if $\kappa(S^\circ) = -\infty$.* Follows immediately from Theorem 7.1 because families over $\mathbb{A}^1$ are necessarily isotrivial by [Kov00, 0.2]. □

## 8. GENERALIZATIONS

**8.1.** It may be worth to note that the proof of Theorem 1.4 uses only the following two facts about families of canonically polarized varieties.

(8.1.1) positive variation guarantees the existence of a sheaf $\mathscr{A} \subset \operatorname{Sym}^n \Omega^1_S(\log D)$ of positive Kodaira-Iitaka dimension—see Theorem 6.1

(8.1.2) families over $\mathbb{P}^1$, $\mathbb{A}^1$, $\mathbb{A}^1 \setminus \{0\}$ and over elliptic curves are necessarily trivial.

Since (8.1.2) is an immediate consequence of (8.1.1), the proof of Theorem 1.4 will work with few modifications whenever we have a family that guarantees the existence of a subsheaf of $\operatorname{Sym}^n \Omega^1_S(\log D)$ similar to what we have in (8.1.1). For instance, [VZ02, Thm. 1.4(iii)] applies to give the following complementary statement to Theorem 1.4.

**Theorem 8.2.** *Let $S^\circ$ be a smooth quasi-projective surface and $f^\circ : X^\circ \to S^\circ$ a smooth family of maximal variation, $\operatorname{Var}(f^\circ) = 2$ such that $\omega_{X^\circ/S^\circ}$ is relatively semi-ample. Then $S^\circ$ is of log general type, i.e., $\kappa(S^\circ) = 2$.* □

*Remark* 8.3. It is conjectured that $\omega_{X^\circ/S^\circ}$ is relatively semi-ample iff $f^\circ$ is a family of minimal manifolds. So far, this is known only if the fibers have dimension at most three.

**Corollary 8.4.** *Viehweg's Conjecture 1.1 holds for families of minimal curves, surfaces or threefolds over surfaces.* □

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STEFAN KEBEKUS, MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, WEYERTAL 86–90, 50931 KÖLN, GERMANY

*E-mail address*: stefan.kebekus@math.uni-koeln.de
*URL*: http://www.mi.uni-koeln.de/∼kebekus

SÁNDOR KOVÁCS, UNIVERSITY OF WASHINGTON, DEPARTMENT OF MATHEMATICS, BOX 354350, SEATTLE, WA 98195, U.S.A.

*E-mail address*: kovacs@math.washington.edu
*URL*: http://www.math.washington.edu/∼kovacs